

GORENSTEIN PROPERTIES OF SIMPLE GLUING ALGEBRAS

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ABSTRACT. Let $A = KQ_A/I_A$ and $B = KQ_B/I_B$ be two finite-dimensional bound quiver algebras, fix two vertices $a \in Q_A$ and $b \in Q_B$. We define an algebra $\Lambda = KQ_\Lambda/I_\Lambda$, which is called a simple gluing algebra of A and B , where Q_Λ is from Q_A and Q_B by identifying a and b , $I_\Lambda = \langle I_A, I_B \rangle$. We prove that Λ is Gorenstein if and only if A and B are Gorenstein, and describe the Gorenstein projective modules, singularity category, Gorenstein defect category and also Cohen-Macaulay Auslander algebra of Λ from the corresponding ones of A and B .

1. INTRODUCTION

In the study of B-branes on Landau-Ginzburg models in the framework of Homological Mirror Symmetry Conjecture, D. Orlov rediscovered the notion of singularity categories [28, 29, 30]. The singularity category of an algebra A is defined to be the Verdier quotient of the bounded derived category with respect to the thick subcategory formed by complexes isomorphic to those consisting of finitely generated projective modules, [10]. It measures the homological singularity of an algebra in the sense that an algebra A has finite global dimension if and only if its singularity category vanishes.

The singularity category captures the stable homological features of an algebra [10]. A fundamental result of R. Buchweitz [10] and D. Happel [21] states that for a Gorenstein algebra A , the singularity category is triangle equivalent to the stable category of Gorenstein projective (also called (maximal) Cohen-Macaulay) A -modules. Buchweitz's Theorem ([10, Theorem 4.4.1]) says that there is an exact embedding $\Phi : \mathbf{Gproj} A \rightarrow D_{sg}(A)$ given by $\Phi(M) = M$, where the second M is the corresponding stalk complex at degree 0, and Φ is an equivalence if and only if A is Gorenstein. Recently, to provide a categorical characterization of Gorenstein algebras, P. A. Bergh, D. A. Jørgensen and S. Oppermann [8] defined the Gorenstein defect category $D_{def}(A) := D_{sg}(A)/\text{Im } \Phi$ and proved that A is Gorenstein if and only if $D_{def}(A) = 0$. In general, it is difficult to describe the singularity categories and Gorenstein defect categories. Many people are trying to describe these categories for some special kinds of algebras, see e.g. [14, 15, 16, 22, 12, 17]. In particular, for a CM-finite algebra A , F. Kong and P. Zhang [24] proved that its Gorenstein defect category is equivalent to the singularity category of its Cohen-Macaulay Auslander algebra. Recently, X-W. Chen [16] described the singularity category and Gorenstein defect category for a quadratic monomial algebra. For a triangular matrix algebra $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, the relations between these categories of Λ and the corresponding ones of A and B are described clearly in some sense [14, 24, 34, 36, 27].

Inspired by the above, we define a new algebra (called *simple gluing algebra*) from some algebras. Explicitly, let $A = KQ_A/I_A$, $B = KQ_B/I_B$ be two finite-dimensional algebras. For any two vertices $a \in Q_A$, $b \in Q_B$, we define a new quiver Q from Q_A and Q_B by identifying a and b . In this way, we can view Q_A and Q_B as subquivers of Q . We call Q the *simple*

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gluing quiver of Q_A and Q_B . Denote by $v \in Q$ the *glued vertex*. Let I be the ideal of KQ generated by I_A and I_B . Then $\Lambda = KQ/I$ is called a simple gluing algebra of A and B if Λ is finite-dimensional. Similar to the triangular matrix algebras, first we prove that Λ is Gorenstein if and only if A and B are Gorenstein, see Proposition 3.4; second we prove that $D_{sg}(\Lambda) \simeq D_{sg}(A) \amalg D_{sg}(B)$ (see Theorem 3.10), $\underline{\text{Gproj}}(\Lambda) \simeq \underline{\text{Gproj}}(A) \amalg \underline{\text{Gproj}}(B)$ (see Theorem 3.14) and $D_{def}(\Lambda) \simeq D_{def}(A) \amalg D_{def}(B)$ (see Corollary 3.16). In particular, if we know the Gorenstein projective modules over A and B , then we can get all the Gorenstein projective modules over Λ . Finally, we prove that the Cohen-Macaulay Auslander algebra of Λ is a simple gluing algebra of the Cohen-Macaulay Auslander algebras of A and B , see Theorem 4.8. As applications, we redescribe the singularity categories for cluster-tilted algebras of type A and endomorphism algebras of maximal rigid objects of cluster tube \mathcal{C}_n .

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2. PRELIMINARY

In this paper, we always assume that K is an algebraically closed field and all algebras are finite-dimensional algebras over K and modules are finitely generated.

Let A be a K -algebra. Let $\text{mod } A$ be the category of finitely generated left A -modules. With $D = \text{Hom}_K(-, K)$ we denote the standard duality with respect to the ground field. Then $D(A_A)$ is an injective cogenerator for $\text{mod } A$. For an arbitrary A -module ${}_A X$ we denote by $\text{proj. dim}_A X$ (resp. $\text{inj. dim}_A X$) the projective dimension (resp. the injective dimension) of the module ${}_A X$.

A complex

$$P^\bullet : \cdots \rightarrow P^{-1} \rightarrow P^0 \xrightarrow{d^0} P^1 \rightarrow \cdots$$

of finitely generated projective A -modules is said to be *totally acyclic* provided it is acyclic and the Hom complex $\text{Hom}_A(P^\bullet, A)$ is also acyclic [6]. An A -module M is said to be (finitely generated) *Gorenstein projective* provided that there is a totally acyclic complex P^\bullet of projective A -modules such that $M \cong \text{Ker } d^0$ [19]. We denote by $\text{Gproj } A$ the full subcategory of $\text{mod } A$ consisting of Gorenstein projective modules.

An algebra is of *finite Cohen-Macaulay type*, or simply, *CM-finite*, if there are only finitely many isomorphism classes of indecomposable finitely generated Gorenstein projective modules. Clearly, A is CM-finite if and only if there is a finitely generated module E such that $\text{Gproj } A = \text{add } E$. In this way, E is called to be a *Gorenstein projective generator*. If A is self-injective, then $\text{Gproj } A = \text{mod } A$, so A is CM-finite if and only if A is representation-finite. An algebra is called *CM-free* if $\text{Gproj } A = \text{proj } A$. If A has finite global dimension, then $\text{Gproj } A = \text{proj } A$, so it is CM-free.

Let A be a CM-finite algebra, E_1, \dots, E_n all the pairwise non-isomorphic indecomposable Gorenstein projective A -modules. Put $E = \bigoplus_{i=1}^n E_i$. Then E is a Gorenstein projective generator. We call $\text{Aus}(\text{Gproj } A) := (\text{End}_A E)^{op}$ the *Cohen-Macaulay Auslander algebra* (also called *relative Auslander algebra*) of A .

Let \mathcal{X} be a subcategory of $\text{mod } A$. Then ${}^\perp \mathcal{X} := \{M \mid \text{Ext}^i(M, X) = 0, \text{ for all } X \in \mathcal{X}, i \geq 1\}$. Dually, we can define \mathcal{X}^\perp . In particular, we define ${}^\perp A := {}^\perp(\text{proj } A)$.

The following lemma follows from the definition of Gorenstein projective module easily.

Lemma 2.1. (i) [7]

$$\begin{aligned} \text{Gproj}(A) = & \{M \in \Lambda\text{-mod} \mid \exists \text{ an exact sequence } 0 \rightarrow M \rightarrow T^0 \xrightarrow{d^0} T^1 \xrightarrow{d^1} \dots, \\ & \text{with } T^i \in \text{proj } A, \ker d^i \in {}^\perp A, \forall i \geq 0\}. \end{aligned}$$

(ii) If M is Gorenstein projective, then $\text{Ext}_A^i(M, L) = 0$, $\forall i > 0$, for all L of finite projective dimension or of finite injective dimension.

(iii) If P^\bullet is a totally acyclic complex, then all $\text{Im } d^i$ are Gorenstein projective; and any truncations

$$\dots \rightarrow P^i \rightarrow \text{Im } d^i \rightarrow 0, \quad 0 \rightarrow \text{Im } d^i \rightarrow P^{i+1} \rightarrow \dots$$

and

$$0 \rightarrow \text{Im } d^i \rightarrow P^{i+1} \rightarrow \dots \rightarrow P^j \rightarrow \text{Im } d^j \rightarrow 0, i < j$$

are $\text{Hom}_A(-, \text{proj } A)$ -exact.

Definition 2.2 ([21], see also [2, 3]). A finite-dimensional algebra A is called a Gorenstein algebra (also called Iwanaga-Gorenstein algebra) if A satisfies $\text{inj. dim } A_A < \infty$ and $\text{inj. dim } {}_A A < \infty$. Given an A -module X . If $\text{Ext}_A^i(X, A) = 0$ for all $i > 0$, then X is called a Cohen-Macaulay module of A .

Observe that for a Gorenstein algebra A , we have $\text{inj. dim } {}_A A = \text{inj. dim } A_A$, see [21, Lemma 6.9]; the common value is denoted by $\text{Gd } A$. If $\text{Gd } A \leq d$, we say that A is d -Gorenstein. Furthermore, since $\text{proj. dim } {}_A D(A_A) = \text{inj. dim } A_A$, we get that A is Gorenstein if and only if $\text{proj. dim } {}_A D(A_A) < \infty$ and $\text{inj. dim } {}_A A < \infty$.

Theorem 2.3 ([10, 19]). Let A be a Gorenstein algebra. Then

(i) If P^\bullet is an exact sequence of projective left A -modules, then $\text{Hom}_A(P^\bullet, A)$ is again an exact sequence of projective right A -modules.

(ii) A module G is Gorenstein projective if and only if there is an exact sequence $0 \rightarrow G \rightarrow P^0 \rightarrow P^1 \rightarrow \dots$ with each P^i projective.

(iii) $\text{Gproj } A = {}^\perp A$.

For a module M , take a short exact sequence

$$0 \rightarrow \Omega M \rightarrow P \rightarrow M \rightarrow 0$$

with P projective. The module ΩM is called a syzygy module of M . For each $i \geq 1$, denote by Ω^i the i -th power of Ω and then for a module X , $\Omega^i X$ is the i -th syzygy module of X . For details, see [4].

Theorem 2.4 ([6]). Let A be an finite-dimensional algebra and $d \geq 0$. Then the following statements are equivalent:

(i) the algebra A is d -Gorenstein;

(ii) $\text{Gproj } A = \Omega^d(\text{mod } A)$.

In this case, we have $\text{Gproj } A = {}^\perp A$.

So for a Gorenstein algebra, the definition of Cohen-Macaulay module coincides with the one of Gorenstein projective.

Recall that for an algebra A , the singularity category of A is the quotient category $D_{sg}(A) := D^b(A)/K^b(\text{proj } A)$, which is defined by Buchweitz [10], see also [21, 28].

Theorem 2.5 (Buchweitz's Theorem, see also [23] for a more general version). Let A be an Artin algebra. Then $\text{Gproj}(A)$ is a Frobenius category with the projective modules as the projective-injective objects, and there is an exact embedding $\Phi : \text{Gproj } A \rightarrow D_{sg}(A)$ given by $\Phi(M) = M$,

where the second M is the corresponding stalk complex at degree 0, and Φ is an equivalence if and only if A is Gorenstein.

Let A be an Artin algebra. Inspired by Buchweitz's Theorem, the *Gorenstein defect category* is defined to be Verdier quotient $D_{def}(A) := D_{sg}(A)/\text{Im}(\Phi)$, see [8]. From [24], we know that $D_{def}(A)$ is triangle equivalent to $D^b(A)/\langle \text{Gproj}(A) \rangle$, where $\langle \text{Gproj}(A) \rangle$ denotes the triangulated subcategory of $D^b(A)$ generated by $\text{Gproj}(A)$, i.e., the smallest triangulated subcategory of $D^b(A)$ containing $\text{Gproj}(A)$.

Lemma 2.6 ([8, 24]). *Let A be an Artin algebra. Then the following are equivalent.*

- (i) A is Gorenstein;
- (ii) $\text{Gproj}(A)$ is triangle equivalent to $D_{sg}(A)$;
- (iii) $D_{def}(A) = 0$;
- (iv) $D^b(A) = \langle \text{Gproj}(A) \rangle$.

3. SIMPLE GLUING ALGEBRAS

Let $Q = (v(Q), a(Q))$ be a finite quiver, where $v(Q)$ is the set of vertices, and $a(Q)$ is the set of arrows. For any arrow α in Q , we denote by $s(\alpha), t(\alpha)$ the source and the target of α respectively. The path algebra KQ of Q is an associative algebra with an identity. If I is an admissible ideal of KQ , the pair (Q, I) is said to be a bound quiver. The quotient algebra KQ/I is said to be the algebra of the bound quiver (Q, I) , or simply, a *bound quiver algebra*. Denote by e_i the idempotent corresponding to the vertex $i \in Q$.

In this paper, we shall freely identify the representations of (Q, I) with left modules over KQ/I . Recall that a representation M of (Q, I) is of form $(M_i, M_\alpha)_{i \in v(Q), \alpha \in a(Q)}$, or (M_i, M_α) . So for any representation M , we always use M_i to denote the vector space associated to the vertex i .

As the beginning, let us recall the definition of *simple gluing algebras*. Let $A = KQ_A/I_A$, $B = KQ_B/I_B$ be two finite-dimensional algebras. For any two vertices $a \in v(Q_A)$, $b \in v(Q_B)$, we define a new quiver Q from Q_A and Q_B by identifying a and b . In this way, we can view Q_A and Q_B as subquivers of Q . We call Q the *simple gluing quiver* of Q_A and Q_B . Denote by $v \in v(Q)$ the *glued vertex*. Let I be the ideal of KQ generated by I_A and I_B .

The following lemma follows directly from the definition of bound quiver algebras.

Lemma 3.1. *Keep the notations as above. Then $\Lambda = KQ/I$ is a finite-dimensional algebra if and only if each non-trivial path in Q_A from a to a is in I_A or each non-trivial path in Q_B from b to b is in I_B .*

In the following, we always assume that the above condition holds, and call Λ the *simple gluing algebra* of A and B . Inductively, from finitely many bound quiver algebras A_1, \dots, A_n , we can define simple gluing algebra (which is finite-dimensional) of A_1, \dots, A_n .

Without loss of generality, we always assume that each non-trivial path in Q_A from a to a is in I_A . Denote by $e_B = \sum_{i \in Q_B} e_i$ the idempotent of Λ . Obviously, A and B are subalgebras (and also quotient algebras) of Λ , and $B \cong e_B \Lambda e_B$.

Since $B = e_B \Lambda e_B$ is a subalgebra of Λ , there exists an exact functor $j_\mu : \text{mod } \Lambda \rightarrow \text{mod } B$, which maps M to $e_B M$. Note that $j_\mu = e_B \Lambda \otimes_\Lambda -$. In fact, j_μ restricts the representation of (Q, I) to the representation of (Q_B, I_B) . j_μ admits a right adjoint functor $j_\rho := \text{Hom}_B(B\Lambda, -) : \text{mod } B \rightarrow \text{mod } \Lambda$, and admits a left adjoint functor $j_\lambda := \Lambda \otimes_B - = \Lambda e_B \otimes_B - : \text{mod } B \rightarrow \text{mod } \Lambda$, see e.g. [18].

Similarly, since A is also a subalgebra of Λ , there exists a natural exact functor $i_\mu : \text{mod } \Lambda \rightarrow \text{mod } A$. i_μ admits a right adjoint functor $i_\rho := \text{Hom}_A(A\Lambda, -) : \text{mod } A \rightarrow \text{mod } \Lambda$, and admits a left adjoint functor $i_\lambda := \Lambda \otimes_A - : \text{mod } A \rightarrow \text{mod } \Lambda$.

From the structure of Λ , obviously, Λ is projective both as left (also right) modules over A and B . So we have the following lemma immediately.

Lemma 3.2. *Keep the notations as above. Then all the functors $j_\lambda, j_\mu, j_\rho, i_\lambda, i_\mu, i_\rho$ are exact functors. In particular, i_λ, i_μ, j_μ and j_λ preserve projectives, i_ρ, i_μ, j_μ and j_ρ preserve injectives.*

For any vertex $i \in v(Q_A)$, we denote by $P(i)$ (resp. ${}_AS(i), I(i)$) the projective (resp. simple, injective) A -module corresponding to i . For any vertex $i \in v(Q_B)$, we denote by $Q(i)$ (resp. ${}_BS(i), J(i)$) the projective (resp. simple, injective) B -module corresponding to i . For any vertex $i \in v(Q)$, we denote by $U(i)$ (resp. ${}_AS(i), V(i)$) the projective (resp. simple, injective) Λ -module corresponding to i .

Note that $i_\lambda(P(i)) = U(i)$, $i_\rho(I(i)) = V(i)$ for any $i \in Q_A$, and $j_\lambda(Q(i)) = U(i)$ and $j_\rho(J(i)) = V(i)$ for any $i \in Q_B$.

In the following, we describe the actions of all these functors on representations. First, from the structure of indecomposable projective modules [1], we can assume that $Q(v)$ is locally as the following left figure shows, where $j_1, \dots, j_q \in Q_A$, $k_1, \dots, k_r \in Q_B$ and each of them is the ending point of some arrow starting at v . Then $j_\lambda({}_BS(b))$ and $i_\lambda({}_AS(a))$ are as the following middle and right figures show respectively.

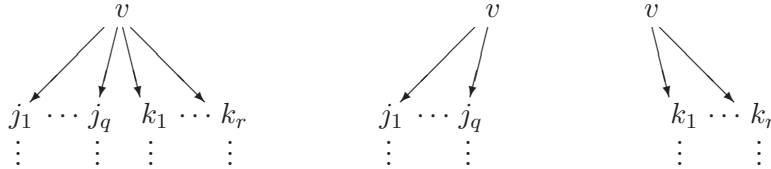


Figure 1. The structures of $Q(v)$, $j_\lambda({}_BS(b))$ and $i_\lambda({}_AS(a))$.

Let $M = (M_i, M_\alpha)$ be a representation of (Q_A, I_A) . We assume that M is locally as the following left figure shows, where i_1, \dots, i_p are the starting points of arrows ending to $v = a$ in Q_A . Then $i_\lambda(M)$ is as the following right figure shows, where the submodule in the dashed box is $i_\lambda({}_AS(a))^{\oplus \dim M_v}$.

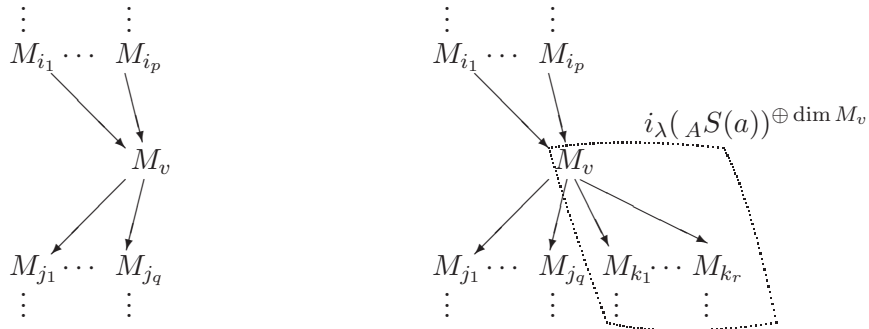


Figure 2. The local structures of $M \in \text{mod } A$ and $i_\lambda(M)$.

The action of j_λ is similar, and i_μ, j_μ are the restriction functors. For i_ρ and j_ρ , their actions are dual to these of i_λ and j_λ .

For any $M \in \text{mod } A$ and $N \in \text{mod } \Lambda$, we denote by

$$\alpha_{M,N} : \text{Hom}_\Lambda(i_\lambda(M), N) \xrightarrow{\sim} \text{Hom}_A(M, i_\mu(N))$$

the adjoint isomorphism. For any $M \in \text{mod } A$, we denote by $\mu_M = \alpha_{M, i_\lambda(M)}(1_{i_\lambda(M)})$ the adjunction morphism, and for any $N \in \text{mod } \Lambda$, we denote by $\epsilon_N = \alpha_{i_\mu(N), N}^{-1}(1_{i_\mu(N)})$.

Similarly, for any $L \in \text{mod } B$ and $N \in \text{mod } \Lambda$ we denote by

$$\beta_{L, N} : \text{Hom}_\Lambda(j_\lambda(L), N) \xrightarrow{\sim} \text{Hom}_B(L, i_\mu(N))$$

the adjoint isomorphism. For any $L \in \text{mod } B$, we denote by $\nu_L = \beta_{L, j_\lambda(L)}(1_{j_\lambda(L)})$ the adjunction morphism, and for any $N \in \text{mod } \Lambda$, we denote by $\zeta_N = \beta_{j_\mu(N), N}^{-1}(1_{j_\mu(N)})$.

It is easy to see that for any $M \in \text{mod } A$, $i_\mu i_\lambda M = M \oplus P(a)^{\oplus m}$ for some m , and μ_M is the section map. Furthermore, for any morphism

$$f = (f_x)_{x \in v(Q_A)} : M = (M_x, M_\alpha)_{x \in v(Q_A), \alpha \in a(Q_A)} \rightarrow N = (N_x, N_\alpha)_{x \in v(Q_A), \alpha \in a(Q_A)}$$

in $\text{mod } A$, we get that $i_\mu i_\lambda(f)$ is of form

$$(1) \quad \begin{pmatrix} f & 0 \\ 0 & h \end{pmatrix} : M \oplus P(a)^{\oplus m} \rightarrow N \oplus P(a)^{\oplus n},$$

where $h : P(a)^{\oplus m} \rightarrow P(a)^{\oplus n}$ can be represented by a $n \times m$ matrix with its entries $h_{ij} : P(a) \rightarrow P(a)$ in K . In particular, h is determined by the map f_v .

For any morphism

$$g = (g_y)_{y \in v(Q_B)} : M = (M_y, M_\beta)_{y \in v(Q_B), \beta \in a(Q_B)} \rightarrow N = (N_y, N_\beta)_{y \in v(Q_B), \beta \in a(Q_B)}$$

in $\text{mod } B$, it is similar to get the form of $j_\mu j_\lambda(g)$. Besides, we also get that $i_\mu j_\lambda(M) = P(a)^{\oplus m}$ for some m , and $i_\mu j_\lambda(N) = P(a)^{\oplus n}$ for some n . Then $i_\mu j_\lambda(g)$ is of form

$$(2) \quad h : P(a)^{\oplus m} \rightarrow P(a)^{\oplus n},$$

where $h : P(a)^{\oplus m} \rightarrow P(a)^{\oplus n}$ can be represented by a $n \times m$ matrix with its entries $h_{ij} : P(a) \rightarrow P(a)$ in K . In particular, h is determined by the map g_v .

Similarly, we can describe $j_\mu i_\lambda(f)$ for any $f : M \rightarrow N$ in $\text{mod } A$.

For any $N \in \text{mod } \Lambda$, we denote by $l(N)$ its length.

Lemma 3.3. *Keep the notations as above. Let $M = (M_i, M_\gamma)$ be a Λ -module. Then we have a short exact sequence*

$$(3) \quad 0 \rightarrow U(v)^{\oplus \dim M_v} \rightarrow i_\lambda i_\mu M \oplus j_\lambda j_\mu M \xrightarrow{(\mu_M, \nu_M)} M \rightarrow 0.$$

Proof. First, suppose that $M = {}_\Lambda S(i)$ is a simple Λ -module. Without loss of generality, we assume that $i \in v(Q_A)$. If $i \neq v$, then $i_\lambda i_\mu({}_\Lambda S(i)) = {}_\Lambda S(i)$ and $j_\lambda j_\mu({}_\Lambda S(i)) = 0$, which satisfies (3). If $i = v$, then $i_\mu({}_\Lambda S(v)) = {}_A S(a)$, and $j_\mu({}_\Lambda S(v)) = {}_B S(a)$. From the structures of $Q(v)$, $j_\lambda({}_B S(b))$ and $i_\lambda({}_A S(a))$ in Figure 1, it is easy to see that there exists a short exact sequence

$$0 \rightarrow U(v) \rightarrow i_\lambda i_\mu S(v) \oplus j_\lambda j_\mu S(v) \rightarrow S(v) \rightarrow 0,$$

which satisfies the requirement.

For general M , we prove it by induction on its length. If $l(M) = 1$, then M is a simple module, and the result follows from the above. If $l(M) > 1$, then there exists M_1, M_2 such that $0 < l(M_i) < l(M)$ for $i = 1, 2$, and there is a short exact sequence

$$0 \rightarrow M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \rightarrow 0.$$

Obviously, $\dim M_v = \dim(M_1)_v + \dim(M_2)_v$ for the glued vertex $v \in v(Q)$.

Since $i_\lambda, i_\mu, j_\lambda, j_\mu$ are exact functors, there are two short exact sequences

$$0 \rightarrow i_\lambda i_\mu M_1 \xrightarrow{i_\lambda i_\mu(f)} i_\lambda i_\mu M \xrightarrow{i_\lambda i_\mu(g)} i_\lambda i_\mu M_2 \rightarrow 0,$$

and

$$0 \rightarrow j_\lambda j_\mu M_1 \xrightarrow{j_\lambda j_\mu(f)} j_\lambda j_\mu M \xrightarrow{j_\lambda j_\mu(g)} j_\lambda j_\mu M_2 \rightarrow 0.$$

The inductive assumption yields the following short exact sequences:

$$0 \rightarrow U(v)^{\oplus \dim(M_1)_v} \rightarrow i_\lambda i_\mu M_1 \oplus j_\lambda j_\mu M_1 \xrightarrow{(\mu_{M_1}, \nu_{M_1})} M_1 \rightarrow 0,$$

and

$$0 \rightarrow U(v)^{\oplus \dim(M_2)_v} \rightarrow i_\lambda i_\mu M_2 \oplus j_\lambda j_\mu M_2 \xrightarrow{(\mu_{M_2}, \nu_{M_2})} M_2 \rightarrow 0.$$

From the naturality of the adjoint pairs (i_λ, i_μ) and (j_λ, j_μ) , we get the following commutative diagram

$$\begin{array}{ccccc} U(v)^{\oplus \dim(M_1)_v} & \longrightarrow & i_\lambda i_\mu M_1 \oplus j_\lambda j_\mu M_1 & \xrightarrow{(\mu_{M_1}, \nu_{M_1})} & M_1 \\ \vdots & & \downarrow \left(\begin{array}{cc} i_\lambda i_\mu(f) & j_\lambda j_\mu(f) \end{array} \right) & & \downarrow f \\ N & \longrightarrow & i_\lambda i_\mu M \oplus j_\lambda j_\mu M & \xrightarrow{(\mu_M, \nu_M)} & M \\ \vdots & & \downarrow \left(\begin{array}{cc} i_\lambda i_\mu(g) & j_\lambda j_\mu(g) \end{array} \right) & & \downarrow g \\ U(v)^{\oplus \dim(M_2)_v} & \longrightarrow & i_\lambda i_\mu M_2 \oplus j_\lambda j_\mu M_2 & \xrightarrow{(\mu_{M_2}, \nu_{M_2})} & M_2 \end{array}$$

where N is the kernel of (μ_M, ν_M) . It is obvious that all the sequences appearing in the rows and columns of the above commutative diagram are short exact sequences. From the short exact sequence in the first column, we get that

$$N \cong U(v)^{\oplus \dim(M_1)_v + \dim(M_2)_v} = U(v)^{\oplus \dim M_v}$$

since $U(v)$ is projective, and then there is a short exact sequence

$$0 \rightarrow U(v)^{\oplus \dim M_v} \rightarrow i_\lambda i_\mu M \oplus j_\lambda j_\mu M \xrightarrow{(\mu_M, \nu_M)} M \rightarrow 0.$$

□

Similar to Gorenstein property of the upper triangular matrix algebras obtained in [14], we get the following result.

Proposition 3.4. *Let Λ be a simple gluing algebra of the two finite-dimensional bound quiver algebras A and B . Then Λ is Gorenstein if and only if A and B are Gorenstein. In particular, $\max\{\text{Gd } A, \text{Gd } B\} \leq \text{Gd }(\Lambda) \leq \max\{\text{Gd } A, \text{Gd } B, 1\}$.*

Proof. Keep the notations as above. If Λ is Gorenstein, then for any indecomposable injective Λ -module $V(i)$, it has finite projective dimension. Let

$$0 \rightarrow U_m \rightarrow \cdots \rightarrow U_1 \rightarrow U_0 \rightarrow V(i) \rightarrow 0$$

be a projective resolution of $V(i)$. Apply i_μ to the above projective resolution. Since i_μ is exact and preserves projectives, we get that $\text{proj. dim}_A(i_\mu(V(i))) \leq m$. It is easy to see that $i_\mu(V(i)) = I(i) \oplus I(a)^{\oplus s}$ for some integer s . Then $\text{proj. dim}_A(I(i)) \leq m$, and $\text{proj. dim}_A D(A) < \infty$. Similarly, we can get that $\text{inj. dim}_A A < \infty$ and so A is Gorenstein. Furthermore, $\text{Gd } A \leq \text{Gd } \Lambda$.

Similarly, B is Gorenstein with $\text{Gd } B \leq \text{Gd } \Lambda$. Then $\max\{\text{Gd } A, \text{Gd } B\} \leq \text{Gd } \Lambda$.

Conversely, for any indecomposable injective Λ -module $V(i)$, $i_\mu(V(i))$ and $j_\mu(V(i))$ are injective as A -module and B -module respectively. Since A and B are Gorenstein, $i_\mu(V(i))$ and $j_\mu(V(i))$ have finite projective dimensions. From i_λ and j_λ are exact and preserve projectives, we get that $i_\lambda i_\mu(V(i))$ and $j_\lambda j_\mu(V(i))$ have finite projective dimensions. In particular,

$$\begin{aligned} \text{proj. dim}_\Lambda i_\lambda i_\mu(V(i)) &\leq \text{proj. dim}_A i_\mu(V(i)) \leq \text{Gd } A \text{ and} \\ \text{proj. dim}_\Lambda j_\lambda j_\mu(V(i)) &\leq \text{proj. dim}_B j_\mu(V(i)) \leq \text{Gd } B. \end{aligned}$$

So $\text{proj. dim}_\Lambda (i_\lambda i_\mu(V(i)) \oplus j_\lambda j_\mu(V(i))) \leq \max\{\text{Gd } A, \text{Gd } B\}$.

From the short exact sequence (3) in Lemma 3.3, it is easy to see that

$$\text{proj. dim}_\Lambda V(i) \leq \max\{\text{Gd } A, \text{Gd } B, 1\}.$$

So $\text{proj. dim}_\Lambda D\Lambda \leq \max\{\text{Gd } A, \text{Gd } B, 1\}$.

Similarly, we can get that

$$\text{inj. dim}_\Lambda \Lambda \leq \max\{\text{Gd } A, \text{Gd } B, 1\}.$$

So Λ is Gorenstein, and $\max\{\text{Gd } A, \text{Gd } B\} \leq \text{Gd}(\Lambda) \leq \max\{\text{Gd } A, \text{Gd } B, 1\}$. \square

From the above Proposition, if A and B are Gorenstein, then we have the following: if $\text{Gd } A = 0 = \text{Gd } B$, then $\text{Gd}(\Lambda) \leq 1$; otherwise, $\text{Gd}(\Lambda) = \max\{\text{Gd } A, \text{Gd } B\}$.

Example 3.5. Let A and B be self-injective (0-Gorenstein) algebras, and Λ be the simple gluing algebra of A and B . In general, Λ is 1-Gorenstein, and is not self-injective. For an example, please see the algebra Λ in Example 4.9.

Example 3.6. For simple gluing algebra $\Lambda = KQ/I$ of A and B , the condition that I is generated by that of I_A and I_B is necessary to ensure the above proposition.

Let Q_Λ be the quiver $1 \begin{matrix} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{matrix} 2 \xleftarrow{\gamma} 3$, Q_A be the full subquiver of Q_Λ containing vertices

1, 2, and Q_B be the full subquiver of Q_Λ containing the vertices 2, 3. Then Q_Λ is a simple gluing quiver of Q_A and Q_B . Let $I_A = \langle \alpha\beta, \beta\alpha \rangle$, $I_B = 0$. Then both A and B are Gorenstein. However, if we take $\Lambda = KQ_\Lambda/I_\Lambda$, where I_Λ is generated by $\alpha\beta, \beta\alpha$ and $\beta\gamma$, then Λ is not Gorenstein any more.

We recall two basic facts concerning functors between triangulated categories, which are useful to prove our main results.

Lemma 3.7 (see. e.g. [15]). Let $F_1 : \mathcal{A} \rightarrow \mathcal{B}$ be an exact functor between abelian categories which has an exact right adjoint F_2 . Then the pair $(D^b(F_1), D^b(F_2))$ is adjoint, where $D^b(F_1)$ is the induced functor from $D^b(\mathcal{A})$ to $D^b(\mathcal{B})$ ($D^b(F_2)$ is defined similarly).

Since all the six functors defined above are exact, so they induces six triangulated functors on derived categories, and $(D^b(i_\lambda), D^b(i_\mu), D^b(i_\rho))$ and $(D^b(j_\lambda), D^b(j_\mu), D^b(j_\rho))$ are adjoint triples. Since $i_\lambda, i_\mu, j_\lambda$ and j_μ preserve projectives, they induce triangulated functors on singularity categories, which are denoted by $\tilde{i}_\lambda, \tilde{i}_\mu, \tilde{j}_\lambda$ and \tilde{j}_μ respectively.

The second one on adjoint functors is well-known.

Lemma 3.8 ([28, 14]). Let \mathcal{M} and \mathcal{N} be full triangulated subcategories in triangulated categories \mathcal{C} and \mathcal{D} respectively. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ be an adjoint pair of exact functors such that $F(\mathcal{M}) \subset \mathcal{N}$ and $G(\mathcal{N}) \subset \mathcal{M}$. Then they induce functors

$$\tilde{F} : \mathcal{C}/\mathcal{M} \rightarrow \mathcal{D}/\mathcal{N}, \quad \tilde{G} : \mathcal{D}/\mathcal{N} \rightarrow \mathcal{C}/\mathcal{M}$$

which are adjoint.

Lemma 3.9. *Keep the notations as above. Then the following hold:*

- (i) $(\tilde{i}_\lambda, \tilde{i}_\mu), (\tilde{j}_\lambda, \tilde{j}_\mu)$ are adjoint pairs;
- (ii) \tilde{i}_λ and \tilde{j}_λ are fully faithful;
- (iii) $\ker(\tilde{j}_\mu) \simeq \text{Im}(\tilde{i}_\lambda)$ and $\ker(\tilde{i}_\mu) \simeq \text{Im}(\tilde{j}_\lambda)$.

Proof. (i) follows from Lemma 3.8 directly.

(ii) we only need to prove that \tilde{i}_λ is fully faithful.

For any A -module M , we have that $i_\mu i_\lambda(M) = M \oplus P(a)^{\oplus s}$ for some integer s . So $\tilde{i}_\mu \tilde{i}_\lambda(M) \cong M$, in particular, this isomorphism is natural which implies that $\tilde{i}_\mu \tilde{i}_\lambda \simeq \text{id}_{D_{sg}(A)}$ since $\text{mod } A$ is a generator of $D_{sg}(A)$. Therefore, \tilde{i}_λ is fully faithful.

(iii) for any A -module M , we have that $j_\mu i_\lambda(M) = Q(b)^{\oplus t}$ for some integer t . So $\tilde{j}_\mu \tilde{i}_\lambda(M) \cong 0$, and then $\tilde{j}_\mu \tilde{i}_\lambda = 0$. It follows that $\text{Im}(\tilde{i}_\lambda) \subseteq \ker(\tilde{j}_\mu)$. Similarly, we can get that $\text{Im}(\tilde{j}_\lambda) \subseteq \ker(\tilde{i}_\mu)$.

For any $M \in \ker(\tilde{j}_\mu)$, the adjunction $\tilde{i}_\lambda \tilde{i}_\mu(M) \xrightarrow{\epsilon_M} M$ is extended to a triangle in $D_{sg}(\Lambda)$:

$$X \rightarrow \tilde{i}_\lambda \tilde{i}_\mu(M) \xrightarrow{\epsilon_M} M \rightarrow X[1]$$

where $[1]$ is the suspension functor. Applying \tilde{i}_μ to the above triangle, we get that $X \in \ker(\tilde{i}_\mu)$ since $\tilde{i}_\mu(\epsilon_M)$ is an isomorphism. Also applying \tilde{j}_μ to the above triangle, we get that $X \in \ker(\tilde{j}_\mu)$ since $M \in \ker(\tilde{j}_\mu)$ and $\tilde{j}_\mu \tilde{i}_\lambda = 0$.

From Lemma 3.3, it is easy to see that $X \cong \tilde{i}_\lambda \tilde{i}_\mu(X) \oplus \tilde{j}_\lambda \tilde{j}_\mu(X)$ in $D_{sg}(\Lambda)$. Together with $X \in \ker(\tilde{i}_\mu) \cap \ker(\tilde{j}_\mu)$, we get that $X \cong 0$ in $D_{sg}(\Lambda)$ and then $M \cong \tilde{i}_\lambda \tilde{i}_\mu(M)$, so $M \in \text{Im}(\tilde{i}_\lambda)$. Therefore, $\ker(\tilde{j}_\mu) \simeq \text{Im}(\tilde{i}_\lambda)$ since \tilde{i}_λ is fully faithful. Similarly, we can get that $\ker(\tilde{i}_\mu) \simeq \text{Im}(\tilde{j}_\lambda)$. \square

It is worth noting that i_λ and j_λ are faithful functors (in general, not full), which can be obtained from the proof of Lemma 3.9.

Theorem 3.10. *Let Λ be a simple gluing algebra of the two finite-dimensional bound quiver algebras A and B . Then $D_{sg}(\Lambda) \simeq D_{sg}(A) \amalg D_{sg}(B)$.*

Proof. For any $L \in D_{sg}(A), N \in D_{sg}(B)$, $\text{Hom}_{D_{sg}(\Lambda)}(\tilde{i}_\lambda(L), \tilde{j}_\lambda(N)) \cong \text{Hom}_{D_{sg}(A)}(L, \tilde{i}_\mu \tilde{j}_\lambda(N)) = 0$ since $\tilde{i}_\mu \tilde{j}_\lambda = 0$. Similarly, $\text{Hom}_{D_{sg}(\Lambda)}(\tilde{j}_\lambda(N), \tilde{i}_\lambda(L)) = 0$.

For any $M \in D_{sg}(\Lambda)$, from Lemma 3.3, we get that $M \cong \tilde{i}_\lambda \tilde{i}_\mu(M) \oplus \tilde{j}_\lambda \tilde{j}_\mu(M)$ in $D_{sg}(\Lambda)$. Together with the fully faithfulness of \tilde{i}_λ and \tilde{j}_λ , we get that $D_{sg}(\Lambda) \simeq \text{Im}(\tilde{i}_\lambda) \amalg \text{Im}(\tilde{j}_\lambda) \simeq D_{sg}(A) \amalg D_{sg}(B)$. \square

Example 3.11. *For simple gluing algebra $\Lambda = KQ_\Lambda/I_\Lambda$ of A and B , the condition that I_Λ is generated by I_A and I_B is necessary to ensure the above theorem.*

Let Q_A, Q_B and Q_Λ be the quivers as the following figure shows. Then Q_Λ is a simple gluing quiver of Q_A and Q_B . Let $A = KQ_A/\langle \varepsilon_1^2 \rangle$, $B = KQ_B/\langle \varepsilon_2^2 \rangle$. If $\Lambda = KQ_\Lambda/I_\Lambda$, where I_Λ is generated by $\varepsilon_1^2, \varepsilon_2^2, \varepsilon_1 \alpha - \alpha \varepsilon_2$, then all these algebras are Gorenstein algebras. It is easy to see that $D_{sg}(A) \simeq D_{sg}(B)$, and $D_{sg}(\Lambda) \simeq D^b(KQ)/[1]$, where Q is a quiver of type \mathbb{A}_2 , and $[1]$ is the suspension functor, see [32]. However, $D_{sg}(\Lambda)$ is not equivalent to $D_{sg}(A) \amalg D_{sg}(B)$.

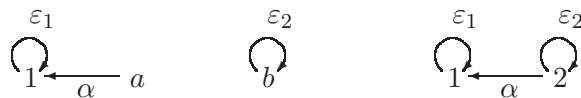


Figure 3. The quivers Q_A , Q_B and Q_Λ in Example 3.11.

Lemma 3.12 ([27]). *Let A_1 and A_2 be Artin algebras. If $F_1 : \text{mod } A_1 \rightarrow \text{mod } A_2$ has the property that it is exact, preserves projective objects, and admits a right adjoint functor F_2 , then*

(i) for any $X \in \text{mod } A_1$ and $Y \in \text{mod } A_2$, we have that $\text{Ext}_{A_2}^k(F_1(X), Y) \cong \text{Ext}_{A_1}^k(X, F_2(Y))$ for any $k \geq 1$.

(ii) if $\text{proj. dim } F_2(Q) < \infty$ or $\text{inj. dim } F_2(Q) < \infty$ for any indecomposable projective A_2 -module Q , then $F_1(\text{Gproj}(A_1)) \subseteq \text{Gproj}(A_2)$.

Lemma 3.13. *Keep the notations as above. Then i_λ , i_μ , j_λ and j_μ preserve Gorenstein projective modules.*

Proof. We only prove that for i_λ and i_μ .

Since (i_λ, i_μ) is an adjoint pair, and both of them are exact and preserve projectives, Lemma 3.12 implies that i_λ preserves Gorenstein projective modules.

For i_μ , first, we claim that $i_\mu M \in {}^\perp A$ for any Gorenstein projective Λ -module M . In fact, for any indecomposable Gorenstein projective Λ -module M , Lemma 3.3 shows that there is a short exact sequence

$$0 \rightarrow U(v)^{\oplus \dim M_v} \rightarrow i_\lambda i_\mu M \oplus j_\lambda j_\mu M \rightarrow M \rightarrow 0,$$

which is split since $M \in {}^\perp \Lambda$, and then $i_\lambda i_\mu M \oplus j_\lambda j_\mu M \cong M \oplus U(v)^{\oplus \dim M_v}$. Note that both $i_\lambda i_\mu M$ and $j_\lambda j_\mu M$ are Gorenstein projective. Since M is indecomposable, we get that $M \in \text{add}(i_\lambda i_\mu M)$ or $M \in \text{add}(j_\lambda j_\mu M)$. If $M \in \text{add}(j_\lambda j_\mu M)$, then $i_\lambda i_\mu M$ is projective, which implies that $i_\mu i_\lambda i_\mu M \in \text{proj } A$. On the other hand, $i_\mu i_\lambda i_\mu M = i_\mu M \oplus P(a)^{\oplus s}$ for some integer s , so $i_\mu M$ is also projective.

If $M \in \text{add}(i_\lambda i_\mu M)$, then for any indecomposable projective Λ -module $U(i)$ with $i \in \mathbf{v}(Q_A)$, Lemma 3.12 shows that

$$\text{Ext}_A^k(i_\mu M, i_\mu U(i)) \cong \text{Ext}_\Lambda^k(i_\lambda i_\mu M, U(i)) = 0$$

for any $k > 0$, since $i_\lambda i_\mu M$ is Gorenstein projective. Furthermore, $i_\mu U(i) = P(i) \oplus P(a)^{\oplus t}$ for some integer t , which implies that $\text{Ext}_\Lambda^k(i_\mu M, P(i)) = 0$ for any $k > 0$ and any indecomposable projective A -module $P(i)$, and then $i_\mu M \in {}^\perp A$.

Second, for any Gorenstein projective Λ -module Z , there is an exact sequence

$$0 \rightarrow Z \rightarrow U_0 \xrightarrow{d_0} U_1 \xrightarrow{d_1} \dots$$

with U_j projective and $\ker(d_j) \in \text{Gproj}(\Lambda)$ for any $j \geq 0$. By applying i_μ to it, we get that there is an exact sequence

$$0 \rightarrow i_\mu Z \rightarrow i_\mu U_0 \xrightarrow{i_\mu d_0} i_\mu U_1 \xrightarrow{i_\mu d_1} \dots$$

with $i_\mu U_j$ projective for any $j \geq 0$ since i_μ is exact and preserves projectives. Additionally, since $\ker(d_i) \in \text{Gproj}(\Lambda)$, from the above, we know that $i_\mu \ker(d_j) \in {}^\perp A$ for any $j \geq 0$, and then $i_\mu Z$ is Gorenstein projective. \square

For any additive category \mathcal{A} , we denote by $\text{Ind } \mathcal{A}$ the set formed by all the indecomposable objects (up to isomorphisms) of \mathcal{A} .

Theorem 3.14. *Let Λ be a simple gluing algebra of the two finite-dimensional bound quiver algebras A and B . Then $\text{Gproj}(\Lambda) \simeq \text{Gproj}(A) \amalg \text{Gproj}(B)$. Furthermore, for any indecomposable Λ -module Z , Z is Gorenstein projective if and only if there exists an indecomposable Gorenstein projective A -module X or B -module Y such that $Z \cong i_\lambda(X)$ or $Z \cong j_\lambda(Y)$.*

Proof. Lemma 3.13 shows that $i_\lambda, i_\mu, j_\lambda, j_\mu$ induce four exact functors on the stable categories of Gorenstein projective modules, which are also denoted by $\tilde{i}_\lambda, \tilde{i}_\mu, \tilde{j}_\lambda, \tilde{j}_\mu$ respectively. Note that \tilde{i}_λ and \tilde{j}_λ are fully faithful.

For any Gorenstein projective Λ -modules Z , Lemma 3.3 yields that $Z \cong \tilde{i}_\lambda \tilde{i}_\mu(Z) \oplus \tilde{j}_\lambda \tilde{j}_\mu(Z)$. Similar to the proof of Theorem 3.10, we get that for any $L \in \underline{\text{Gproj}}(A), N \in \underline{\text{Gproj}}(B)$, $\text{Hom}_{\underline{\text{Gproj}}(\Lambda)}(\tilde{i}_\lambda(L), \tilde{j}_\lambda(N)) = 0 = \text{Hom}_{\underline{\text{Gproj}}(\Lambda)}(\tilde{j}_\lambda(N), \tilde{i}_\lambda(L))$. Therefore,

$$\underline{\text{Gproj}}(\Lambda) \simeq \underline{\text{Gproj}}(A) \amalg \underline{\text{Gproj}}(B).$$

Obviously, i_λ, j_λ preserve indecomposable modules. So for any indecomposable Gorenstein projective A -module X and B -module Y , we get that $i_\lambda(X), j_\lambda(Y)$ are indecomposable Gorenstein projective Λ -modules. On the other hand, for any indecomposable Gorenstein projective Λ -module Z , if Z is projective, then $Z = U(i)$ for some vertex i . If $i \in v(Q_A)$, then $Z = i_\lambda(P(i))$; if $i \in v(Q_B)$, then $Z = j_\lambda(Q(i))$.

If Z is not projective, then the proof of Lemma 3.13 implies that one and only one of $i_\mu Z, j_\mu Z$ is not projective. If $i_\mu Z$ is not projective, then $Z \cong i_\lambda i_\mu Z$ in $\underline{\text{Gproj}}(\Lambda)$. Set $i_\mu Z = \bigoplus_{i=1}^n Z_i$, where Z_i is indecomposable for each $1 \leq i \leq n$. Then $i_\lambda i_\mu Z = \bigoplus_{i=1}^n i_\lambda(Z_i)$ which is isomorphic to Z in $\underline{\text{Gproj}}(\Lambda)$. So there is one and only one non-projective Z_{i_0} such that $i_\lambda(Z_{i_0}) \cong Z$. Since $i_\mu(Z) \in \underline{\text{Gproj}}(A)$, we get that Z_{i_0} is an indecomposable Gorenstein projective A -module such that $Z \cong i_\lambda(Z_{i_0})$.

If $j_\mu Z$ is not projective, we can prove it similarly. \square

Corollary 3.15. *Let Λ be a simple gluing algebra of several finite-dimensional bound quiver algebras A_1, \dots, A_n . If A_1, \dots, A_n are self-injective algebras, then*

$$D_{sg}(\Lambda) \simeq \underline{\text{Gproj}}\Lambda \simeq \coprod_{i=1}^n \underline{\text{mod}}A_i.$$

Proof. It follows from Proposition 3.4 and Theorem 3.14 immediately. \square

Corollary 3.16. *Let Λ be a simple gluing algebra of the two finite-dimensional bound quiver algebras A and B . Then $D_{def}(\Lambda) \simeq D_{def}(A) \amalg D_{def}(B)$.*

Proof. From Theorem 3.10 and Theorem 3.14, we get that

$$D_{sg}(\Lambda) \simeq D_{sg}(A) \amalg D_{sg}(B), \text{ and } \underline{\text{Gproj}}(\Lambda) \simeq \underline{\text{Gproj}}(A) \amalg \underline{\text{Gproj}}(B).$$

In particular, since these two equivalences are induced by the functors i_λ, j_λ , they are compatible, which implies the result by the definition of Gorenstein defect categories immediately. \square

Corollary 3.17. *Keep the notations as above. Then for any $X \in \underline{\text{Gproj}}(A), Y \in \underline{\text{Gproj}}(B)$, and any morphism $f \in \text{Hom}_\Lambda(i_\lambda(X), j_\lambda(Y))$, we have that f factors through $U(v)^{\oplus n}$ for some n .*

Proof. From Theorem 3.14, we get that f factors through some projective Λ -module U as $f = f_2 f_1$, where $f_1 : i_\lambda(X) \rightarrow U, f_2 : U \rightarrow j_\lambda(Y)$. Let $Q_Y \xrightarrow{\alpha_1} Y$ be the projective cover of Y . Then f_2 factors through $j_\lambda(Q_Y)$ as $f_2 = j_\lambda(\alpha_1) f_3$ for some $f_3 : U \rightarrow j_\lambda(Q_Y)$. Since X is Gorenstein projective, there is a short exact sequence

$$0 \rightarrow X \xrightarrow{\beta_1} P_X \xrightarrow{\beta_2} X_1 \rightarrow 0$$

such that P_X projective, and $X_1 \in \underline{\text{Gproj}}(A)$. Then f_1 factors through $i_\lambda(P_X)$ as $f_1 = f_4 i_\lambda(\beta_1)$ for some $f_4 : i_\lambda(P_X) \rightarrow U$ since i_λ is exact and preserves Gorenstein projective modules. So $f = j_\lambda(\alpha_1) f_3 f_4 i_\lambda(\beta_1)$. For any indecomposable Λ -modules $U(i), U(j)$ with $i \in v(Q_A), j \in v(Q_B)$, and any morphism $\gamma : U(i) \rightarrow U(j)$, it is easy to see that γ factors through some object in $\text{add } U(v)$. So $f_3 f_4 : i_\lambda(P_X) \rightarrow j_\lambda(Q_Y)$ factors through $U(v)^{\oplus n}$ for some n , and then the desired result follows. \square

Similarly, for any $X \in \text{Gproj}(A)$, $Y \in \text{Gproj}(B)$, and any morphism $g \in \text{Hom}_\Lambda(j_\lambda(Y), i_\lambda(X))$, then g factors through $U(v)^{\oplus n}$ for some n .

Example 3.18. For simple gluing algebra $\Lambda = KQ/I$ of A and B , the condition that I is generated by I_A and I_B is necessary to ensure that the above corollary holds.

Keep the notations as in Example 3.6. Then $D_{\text{def}}(A) = 0 = D_{\text{def}}(B)$ since A and B are Gorenstein. However, $D_{\text{def}}(\Lambda) \neq 0$ since Λ is not Gorenstein.

The following example is taken from [16].

Example 3.19. Let $\Lambda = KQ/I$ be the algebra where Q is the quiver $1 \begin{smallmatrix} \xrightarrow{\alpha_1} \\ \xleftarrow{\beta_1} \end{smallmatrix} 2 \begin{smallmatrix} \xrightarrow{\alpha_2} \\ \xleftarrow{\beta_2} \end{smallmatrix} 3 \xleftarrow{\gamma} 4$,

and $I = \langle \beta_i \alpha_i, \alpha_i \beta_i, \beta_2 \gamma \mid i = 1, 2 \rangle$. Let $A = (e_1 + e_2)\Lambda(e_1 + e_2)$, and $B = (1 - e_1)\Lambda(1 - e_1)$. Then Λ is a simple gluing algebra of A and B . Obviously, A is self-injective with the indecomposable non-projective Gorenstein projective modules ${}_A S(1)$, ${}_A S(2)$; B is CM-free, and then $D_{\text{def}}(B) = D_{\text{sg}}(B)$. Theorem 3.10 shows that $D_{\text{sg}}(\Lambda) \simeq D_{\text{sg}}(A) \amalg D_{\text{sg}}(B)$. From Theorem 3.14, we get that $\underline{\text{Gproj}}(\Lambda) \simeq \underline{\text{Gproj}}(A) = \underline{\text{mod}} A$. In particular, the indecomposable non-projective Gorenstein projective Λ -modules are $i_\lambda({}_A S_1) = {}_\Lambda S_1$, $i_\lambda({}_A S_2) = \text{rad } U(1)$ (which is the string module with its string $2 \xrightarrow{\alpha_2} 3$). From Corollary 3.16, we get that $D_{\text{def}}(\Lambda) \simeq D_{\text{sg}}(B)$.

The following corollary follows from Theorem 3.14 directly.

Corollary 3.20. Let Λ be a simple gluing algebra of the two finite-dimensional bound quiver algebras $A = KQ_A/I_A$ and $B = KQ_B/I_B$. Then

- (i) Λ is CM-free if and only if A and B are CM-free;
- (ii) Λ is CM-finite if and only if A and B are CM-finite.

At the end of this section, we describe the singularity categories of cluster-tilted algebras of type \mathbb{A}_n and endomorphism algebras of maximal rigid objects of cluster tube \mathcal{C}_n . Note that both of them are 1-Gorenstein algebras.

Let $H_1 = K(\circ \rightarrow \circ)$ be the hereditary algebra of type \mathbb{A}_2 , $H_2 = KQ/I$ the self-injective algebra, where Q is the quiver

$$\begin{array}{ccc} & \circ & \\ \alpha_1 \nearrow & & \searrow \alpha_2 \\ \circ & & \circ \\ & \xleftarrow{\alpha_3} & \end{array} \quad \text{and } I = \langle \alpha_2 \alpha_1, \alpha_3 \alpha_2, \alpha_1 \alpha_3 \rangle.$$

Let A_1, \dots, A_n be algebras where each of them is either H_1 or H_2 . The cluster-tilted algebra of type \mathbb{A} is an algebra Λ which is a simple gluing algebra of A_1, \dots, A_n such that there are only two of A_1, \dots, A_n glued at each glued vertex, see [9]. From Theorem 3.14, we get the following result immediately.

Corollary 3.21 ([12, 22]). Let KQ/I be a cluster-tilted algebra of type \mathbb{A} . Then

$$\underline{\text{Gproj}}(KQ/I) \simeq \coprod_{t(Q)} \underline{\text{mod}} H_2,$$

where $t(Q)$ is the number of the oriented cycles of length 3 in Q .

Let KQ_1/I_1 be a cluster-tilted algebra of type \mathbb{A}_{n-1} ($n \geq 2$), and $KQ_2/\langle \varphi^2 \rangle$, where Q_2 is a loop φ . Note that KQ_1/I_1 is a simple gluing algebra of several algebras which are isomorphic to H_1 or H_2 . Choose a vertex $a \in Q_1$ which is not a glued vertex for Q_1 . Then define KQ/I to be the simple gluing algebra of KQ_1/I_1 and KQ_2/I_2 by identifying a and the unique vertex of

Q_2 . From [33, 35], we get that KQ/I is the endomorphism algebra of a maximal rigid object of cluster tube \mathcal{C}_n , and all endomorphism algebras of maximal rigid objects of cluster tube \mathcal{C}_n are arisen in this way. Theorem 3.14 yields the following result immediately.

Corollary 3.22 ([22]). *Let KQ/I be an endomorphism algebra of a maximal rigid objects of cluster tube \mathcal{C}_n . Then*

$$\underline{\text{Gproj}}(KQ/I) \simeq \coprod_{t(Q)} \underline{\text{mod}} H_2 \coprod \underline{\text{mod}} K[X]/\langle X^2 \rangle.$$

where $t(Q)$ is the number of the oriented cycles of length 3 in Q .

4. COHEN-MACAULAY AUSLANDER ALGEBRAS OF SIMPLE GLUING ALGEBRAS

In this section, we describe the Cohen-Macaulay Auslander algebras for CM-finite simple gluing algebras.

First, let us recall some results about almost split sequences and irreducible morphisms.

Lemma 4.1 (see e.g. [26]). *Let \mathcal{A} be a Krull-Schmidt exact K -category with $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ an almost split sequence.*

(i) *Up to isomorphism, the sequence is the unique almost split sequence starting with X and the unique one ending with Z .*

(ii) *Each irreducible morphism $f_1 : X \rightarrow Y_1$ or $g_1 : Y_1 \rightarrow Z$ fits into an almost split sequence*

$$0 \rightarrow X \xrightarrow{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}} Y_1 \oplus Y_2 \xrightarrow{(g_1, g_2)} Z \rightarrow 0.$$

Recall that for a CM-finite algebra Λ , $\text{Gproj } \Lambda$ is a functorially finite subcategory of $\text{mod } \Lambda$, which implies that $\text{Gproj } \Lambda$ has almost split sequences, see [5, Theorem 2.4], and then $\underline{\text{Gproj}} \Lambda$ has Auslander-Reiten triangles. $\text{Aus}(\text{Gproj } \Lambda)$ is isomorphic to the opposite algebra of $KQ^{\text{Aus}}/I^{\text{Aus}}$ for some ideal I^{Aus} , see Chapter VII, Section 2 of [4]. For any irreducible morphism $f : U(i) \rightarrow U(j)$ in $\text{Gproj } \Lambda$, then it is irreducible in $\text{proj } \Lambda$, and there exists some arrow $\alpha : j \rightarrow i$ such that f is induced by α .

In the following, we always assume that the simple gluing algebra Λ is CM-finite. Since Λ is CM-finite, Corollary 3.20 yields that both A and B are CM-finite. Let Q_A^{Aus} , Q_B^{Aus} and Q_Λ^{Aus} be the Auslander-Reiten quivers of $\text{Gproj } A$, $\text{Gproj } B$ and $\text{Gproj } \Lambda$ respectively. First, we prove that Q_Λ^{Aus} is a simple gluing quiver of Q_A^{Aus} and Q_B^{Aus} by identifying the vertices corresponding to the indecomposable projective modules $P(a)$ and $Q(b)$. Before that, we give two lemmas.

Lemma 4.2. *Let $\Lambda = KQ_\Lambda/I_\Lambda$ be a simple gluing algebra of the two finite-dimensional bound quiver algebras $A = KQ_A/I_A$ and $B = KQ_B/I_B$ by identifying $a \in Q_A$ and $b \in Q_B$. Then for any vertices i, j in Q_Λ and any arrow $\alpha : j \rightarrow i$ in Q_Λ , we have*

(i) *if α is in Q_A , and $f : P(i) \rightarrow P(j)$ is the irreducible morphism in $\text{Gproj } A$ induced by the arrow α , then $i_\lambda(f) : U(i) \rightarrow U(j)$ is irreducible.*

(ii) *if α is in Q_B , and $g : P(i) \rightarrow P(j)$ is the irreducible morphism in $\text{Gproj } B$ induced by the arrow α , then $j_\lambda(g) : U(i) \rightarrow U(j)$ is irreducible.*

(iii) *any irreducible morphism from $U(i)$ to $U(j)$ in $\text{Gproj } \Lambda$ is of form $i_\lambda(f)$ for some irreducible morphism $f : P(i) \rightarrow P(j)$ in $\text{Gproj } A$ or $j_\lambda(g)$ for some irreducible morphism $g : Q(i) \rightarrow Q(j)$ in $\text{Gproj } B$.*

Proof. (i) if $i_\lambda(f)$ factors through an object in $\text{Gproj}(\Lambda)$, then by Theorem 3.14, it is of form $i_\lambda(f) = hg$ for some $i_\lambda(P(i)) \xrightarrow{g} i_\lambda(L) \oplus j_\lambda(M) \xrightarrow{h} i_\lambda(P(j))$, where $L \in \text{Gproj}(A)$, and $M \in \text{Gproj}(B)$. We only need to prove that either g is a section or h is a retraction.

Since f is induced by an arrow $\alpha : j \rightarrow i$ in Q_A , it is easy to see that $i_\lambda(f) : U(i) \rightarrow U(j)$ is the morphism induced by the arrow $\alpha : j \rightarrow i$ in Q_Λ , which is irreducible in $\text{proj } \Lambda$. We assume that $L = L_p \oplus L_g$ where L_p is projective, L_g satisfies that all its indecomposable direct summands are non-projective, and $M = M_p \oplus M_g$ where M_p is projective, M_g satisfies that all its indecomposable direct summands are non-projective. So there exist the following exact sequences

$$0 \rightarrow L'_g \xrightarrow{k_0} P_0 \xrightarrow{k_1} L_g \rightarrow 0, \text{ and } 0 \rightarrow L_g \xrightarrow{l_0} P_1 \xrightarrow{l_1} L''_g \rightarrow 0,$$

with $P_0, P_1 \in \text{proj } A$ and $L'_g, L''_g \in \text{Gproj } A$. Similarly, there are exact sequences

$$0 \rightarrow M'_g \xrightarrow{s_0} Q_0 \xrightarrow{s_1} M_g \rightarrow 0, \text{ and } 0 \rightarrow M_g \xrightarrow{t_0} Q_1 \xrightarrow{t_1} M''_g \rightarrow 0,$$

with $Q_0, Q_1 \in \text{proj } B$ and $M'_g, M''_g \in \text{Gproj } B$. Since i_λ and j_λ are exact, preserve projective modules, there exist the following short exact sequences

$$\begin{aligned} 0 \rightarrow i_\lambda(L'_g) \oplus j_\lambda(M'_g) &\xrightarrow{\begin{pmatrix} 0 & 0 \\ i_\lambda(k_0) & 0 \\ 0 & 0 \\ 0 & j_\lambda(s_0) \end{pmatrix}} i_\lambda(L_p) \oplus i_\lambda(P_0) \oplus j_\lambda(M_p) \oplus j_\lambda(Q_0) \\ &\xrightarrow{\begin{pmatrix} 1 & & & \\ & i_\lambda(k_1) & & \\ & & 1 & \\ & & & j_\lambda(s_1) \end{pmatrix}} i_\lambda(L_p) \oplus i_\lambda(L_g) \oplus j_\lambda(M_p) \oplus j_\lambda(M_g) \rightarrow 0, \end{aligned}$$

and

$$\begin{aligned} 0 \rightarrow i_\lambda(L_p) \oplus i_\lambda(L_g) \oplus j_\lambda(M_p) \oplus j_\lambda(M_g) &\xrightarrow{\begin{pmatrix} 1 & & & \\ & i_\lambda(l_0) & & \\ & & 1 & \\ & & & j_\lambda(t_0) \end{pmatrix}} \\ i_\lambda(L_p) \oplus i_\lambda(P_1) \oplus j_\lambda(M_p) \oplus j_\lambda(Q_1) &\xrightarrow{\begin{pmatrix} 0 & i_\lambda(l_1) & 0 & 0 \\ 0 & 0 & 0 & j_\lambda(t_1) \end{pmatrix}} i_\lambda(L''_g) \oplus j_\lambda(M''_g) \rightarrow 0. \end{aligned}$$

It is easy to see that there exist morphisms $p : i_\lambda(P(i)) \rightarrow i_\lambda(L_p) \oplus i_\lambda(P_0) \oplus j_\lambda(M_p) \oplus j_\lambda(Q_0)$ and $q : i_\lambda(L_p) \oplus i_\lambda(P_1) \oplus j_\lambda(M_p) \oplus j_\lambda(Q_1) \rightarrow i_\lambda(P(i))$ such that

$$g = \begin{pmatrix} 1 & & & \\ & i_\lambda(k_1) & & \\ & & 1 & \\ & & & j_\lambda(s_1) \end{pmatrix} p, \text{ and } h = q \begin{pmatrix} 1 & & & \\ & i_\lambda(l_0) & & \\ & & 1 & \\ & & & j_\lambda(t_0) \end{pmatrix}.$$

Therefore,

$$i_\lambda(f) = hg = q \begin{pmatrix} 1 & & & \\ & i_\lambda(l_0 k_1) & & \\ & & 1 & \\ & & & j_\lambda(t_0 s_1) \end{pmatrix} p.$$

Since $i_\lambda(f)$ is irreducible in $\text{proj } \Lambda$, we get that either p is a section or q is a retraction.

For p is a section, if p induces $i_\lambda(P(i))$ to be a direct summand of $i_\lambda(L_p)$ or a direct summand of $j_\lambda(M_p)$, it is easy to see that g is a section. If p induces $i_\lambda(P(i))$ to be a direct summand of $i_\lambda(P_g)$, then

$$\begin{pmatrix} 1 & & & \\ & i_\lambda(l_0 k_1) & & \\ & & 1 & \\ & & & j_\lambda(t_0 s_1) \end{pmatrix} p$$

is not a section, which yields that q is a retraction, in particular, q induces that $i_\lambda(P(j))$ is a direct summand of $i_\lambda(P_1)$. So f factors through L_g which implies that f is not irreducible in $\text{Gproj } A$ since L_g satisfies that all its indecomposable direct summands are non-projective, a contradiction. If p induces $i_\lambda(P(i))$ to be a direct summand of $j_\lambda(M_g)$, then we get that $i_\lambda(f)$ factors through $j_\lambda(t_0s_1)$. From $t_0s_1 : Q_0 \rightarrow Q_1$ factors through M_g , it is easy to see that there exists an element $w = \sum_{i=0}^n d_i w_i \in KQ_\Lambda$ with $d_i \in K$, w_i a path containing as least one arrow in Q_B for each $1 \leq i \leq n$, such that $\alpha - w \in I_\Lambda$, which is impossible.

For q is a retraction, it is dual to the above, we omit the proof here. So $i_\lambda(f)$ is irreducible.

For (ii), it is similar to (i).

(iii) for any irreducible morphism ψ from $U(i)$ to $U(j)$ in $\text{Gproj } \Lambda$, then it is also irreducible in $\text{proj } \Lambda$, which is induced by an arrow $\alpha : j \rightarrow i$. If α is in Q_A , then $\psi = i_\lambda(f)$ where f is the irreducible morphism $f : P(i) \rightarrow P(j)$ in $\text{proj } A$ induced by α . Suppose for a contradiction that there exist morphisms f_1 and f_2 in $\text{Gproj } A$ such that $f = f_2 f_1$ with neither f_1 a section nor f_2 a retraction. Then it is easy to see that neither $i_\lambda(f_1)$ is a section nor $i_\lambda(f_2)$ a retraction, and $i_\lambda(f) = i_\lambda(f_2) i_\lambda(f_1)$, a contradiction to that $i_\lambda(f)$ is irreducible in $\text{Gproj } \Lambda$.

If α is in Q_B , then we can prove it similarly. \square

Lemma 4.3. *Let $\Lambda = KQ_\Lambda/I_\Lambda$ be a simple gluing algebra of the two finite-dimensional bound quiver algebras $A = KQ_A/I_A$ and $B = KQ_B/I_B$ by identifying $a \in Q_A$ and $b \in Q_B$. If Λ is CM-finite, then*

(i) *for any irreducible morphism $f : X \rightarrow Y$ in $\text{Gproj } A$ with either X or Y not projective, we have that $i_\lambda(f)$ is irreducible in $\text{Gproj } \Lambda$;*

(ii) *for any irreducible morphism $g : L \rightarrow M$ in $\text{Gproj } B$ with either L or M not projective, we have that $j_\lambda(g)$ is irreducible in $\text{Gproj } \Lambda$;*

(iii) *any irreducible morphism in $\text{Gproj } \Lambda$ is of form $i_\lambda(f)$ for some irreducible morphism $f : X \rightarrow Y$ in $\text{Gproj } A$ or $j_\lambda(g)$ for some irreducible morphism $g : L \rightarrow M$ in $\text{Gproj } B$.*

Proof. Since Λ is CM-finite, Corollary 3.20 yields that both A and B are CM-finite. Then all the categories $\text{Gproj } A$, $\text{Gproj } B$ and $\text{Gproj } \Lambda$ have almost split sequences.

First, we prove that i_λ preserves almost split sequences. For any almost split sequence

$$0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$$

in $\text{Gproj}(\Lambda)$, obviously, L and N are indecomposable, and

$$L \xrightarrow{\tilde{f}} M \xrightarrow{\tilde{g}} N \rightarrow \Sigma L$$

is an Auslander-Reiten triangle in $\text{Gproj}(\Lambda)$, where Σ is its suspension functor. Obviously, $i_\lambda(L)$ and $i_\lambda(N)$ are indecomposable modules. Since i_λ is exact, the sequence

$$(4) \quad 0 \rightarrow i_\lambda(L) \xrightarrow{i_\lambda(f)} i_\lambda(M) \xrightarrow{i_\lambda(g)} i_\lambda(N) \rightarrow 0$$

is exact. We claim that the sequence (4) is almost split. From Theorem 3.14, we get that \tilde{i}_λ preserves Auslander-Reiten triangles, and then

$$(5) \quad i_\lambda(L) \xrightarrow{\widetilde{i_\lambda(f)}} i_\lambda(M) \xrightarrow{\widetilde{i_\lambda(g)}} i_\lambda(N) \rightarrow \Sigma L$$

is an Auslander-Reiten triangles in $\text{Gproj}(\Lambda)$. For any morphism $h : Z \rightarrow i_\lambda(N)$ which is not a retraction, then in $\text{Gproj}(\Lambda)$, \tilde{h} is also not a retraction. The Auslander-Reiten triangle (5) yields that \tilde{h} factors through $\widetilde{i_\lambda(g)}$ as $\tilde{h} = \widetilde{i_\lambda(g)} \tilde{l}$ for some morphism $l : Z \rightarrow i_\lambda(M)$. Then there

exists a projective Λ -module U such that $h = i_\lambda(g)l + p_2p_1$ for some morphism $p_1 : Z \rightarrow U$ and $p_2 : U \rightarrow i_\lambda(N)$ as the following diagram shows

$$\begin{array}{ccccc} i_\lambda(L) & \xrightarrow{i_\lambda(f)} & i_\lambda(M) & \xrightarrow{i_\lambda(g)} & i_\lambda(N) \\ & & \uparrow l & \nearrow h & \uparrow p_2 \\ & & Z & \xrightarrow{p_1} & U. \end{array}$$

Since U is projective, we get that p_2 factors through $i_\lambda(g)$ as $p_2i_\lambda(g)p_3$ for some morphism $p_3 : U \rightarrow i_\lambda(M)$. Then $h = i_\lambda(g)l + p_2p_1 = i_\lambda(g)(l + p_3p_1)$ which implies that $i_\lambda(g)$ is a right almost split morphism. Therefore, the sequence (4) is almost split in $\text{Gproj } \Lambda$.

Similarly, we get that j_λ preserves almost split sequences.

Theorem 3.14 also implies that for any Auslander-Reiten triangles in $\underline{\text{Gproj}} \Lambda$, this triangle is either from an Auslander-Reiten triangle in $\underline{\text{Gproj}}(A)$ mapped by \tilde{i}_λ or from an Auslander-Reiten triangle in $\underline{\text{Gproj}}(B)$ mapped by \tilde{j}_λ . Note that all the Auslander-Reiten triangles in the stable categories are induced by almost split sequences. So any almost split sequence in $\text{Gproj}(\Lambda)$ is either of form

$$0 \rightarrow i_\lambda(L) \xrightarrow{i_\lambda(u)} i_\lambda(M) \xrightarrow{i_\lambda(v)} i_\lambda(N) \rightarrow 0$$

where $0 \rightarrow L \xrightarrow{u} M \xrightarrow{v} N \rightarrow 0$ is an almost split sequence in $\text{Gproj}(A)$; or of form

$$0 \rightarrow j_\lambda(X) \xrightarrow{j_\lambda(u)} j_\lambda(Y) \xrightarrow{j_\lambda(v)} j_\lambda(Z) \rightarrow 0$$

where $0 \rightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \rightarrow 0$ is an almost split sequence in $\text{Gproj}(B)$.

(i) for any irreducible morphism $f : X \rightarrow Y$ in $\text{Gproj } A$, if X is not projective, then there is an almost split sequence starting with X , which is of form

$$0 \rightarrow X \xrightarrow{\begin{pmatrix} f \\ f_2 \end{pmatrix}} Y_1 \oplus Y_2 \xrightarrow{(g_1, g_2)} Z \rightarrow 0.$$

by Lemma 4.1 (ii). From above, we get an almost split sequence in $\text{Gproj } \Lambda$

$$0 \rightarrow i_\lambda X \xrightarrow{\begin{pmatrix} i_\lambda(f) \\ i_\lambda(f_2) \end{pmatrix}} i_\lambda(Y_1) \oplus i_\lambda(Y_2) \xrightarrow{(i_\lambda(g_1), i_\lambda(g_2))} i_\lambda Z \rightarrow 0.$$

From it, it is easy to see that $i_\lambda(f)$ is irreducible. If Y is not projective, we can prove it similarly.

(ii) is similar to (i).

(iii) for any irreducible morphism $\psi : W \rightarrow Z$ in $\text{Gproj } \Lambda$, if $W, Z \in \text{proj } \Lambda$, then it follows from Lemma 4.2 (iii) immediately. Otherwise, if W is not projective, then Theorem 3.14 shows that W is of form $i_\lambda(X)$ or $j_\lambda(L)$ for some non-projective indecomposable Gorenstein projective modules $X \in \text{Gproj } A$ and $L \in \text{Gproj } B$. We only prove for the case $W = i_\lambda(X)$, since the other one is similar. Then there is an almost split sequence starting with $i_\lambda(X)$, by the above, it is of form

$$0 \rightarrow i_\lambda(X) \xrightarrow{i_\lambda(f_1)} i_\lambda(Y_1) \xrightarrow{i_\lambda(g_1)} i_\lambda(Z_1) \rightarrow 0.$$

From Lemma 4.1 (ii), it is easy to see that $\psi = i_\lambda(f)$ for some $i_\lambda(f) : i_\lambda(X) \rightarrow i_\lambda(Y)$, where Y is an indecomposable direct summand of Y_1 . Since $i_\lambda(f)$ is irreducible, similar to the proof of Lemma 4.2 (iii), we get that f is irreducible in $\text{Gproj } A$.

For the case when Z is not projective, we can prove it similarly. \square

Recall that for any CM-finite algebra Λ , the Auslander-Reiten quiver of $\text{Gproj } \Lambda$ is formed by indecomposable objects and irreducible morphisms in $\text{Gproj } \Lambda$. So we get the following result by Lemma 4.2 and Lemma 4.3 immediately.

Proposition 4.4. *Let $\Lambda = KQ_\Lambda/I_\Lambda$ be a simple gluing algebra of the two finite-dimensional bound quiver algebras $A = KQ_A/I_A$ and $B = KQ_B/I_B$ by identifying $a \in Q_A$ and $b \in Q_B$. Then Λ is CM-finite if and only if A and B are CM-finite. In this case, the Auslander-Reiten quiver Q_Λ^{Aus} of Λ is a simple gluing quiver of the Auslander-Reiten quiver Q_A^{Aus} of A and the Auslander-Reiten quiver Q_B^{Aus} of B by identifying the vertices corresponding to the indecomposable projective modules $P(a)$ and $Q(b)$.*

Recall that for any $M \in \text{mod } A$ and $N \in \text{mod } \Lambda$, we denote by

$$\alpha_{M,N} : \text{Hom}_\Lambda(i_\lambda(M), N) \xrightarrow{\sim} \text{Hom}_A(M, i_\mu(N))$$

the adjoint isomorphism. For any $M \in \text{mod } A$, we denote by $\mu_M = \alpha_{M, i_\lambda(M)}(1_{i_\lambda(M)})$ the adjunction morphism. For any $L \in \text{mod } B$ and $N \in \text{mod } \Lambda$ we denote by

$$\beta_{L,N} : \text{Hom}_\Lambda(j_\lambda(L), N) \xrightarrow{\sim} \text{Hom}_B(L, i_\mu(N))$$

the adjoint isomorphism.

Lemma 4.5. *Let $\Lambda = KQ_\Lambda/I_\Lambda$ be a simple gluing algebra of the two finite-dimensional bound quiver algebras $A = KQ_A/I_A$ and $B = KQ_B/I_B$ by identifying $a \in Q_A$ and $b \in Q_B$. Let $X \in \text{Gproj } A$ and $Y \in \text{Gproj } B$ be indecomposable Gorenstein projective modules. For any morphisms $f_i : X \rightarrow P(a)$ and $g_i : Q(b) \rightarrow Y$ where $1 \leq i \leq n$, if f_1, \dots, f_n are linearly independent and $\sum_{i=1}^n j_\lambda(g_i)i_\lambda(f_i) = 0$, then $g_i = 0$ for any $1 \leq i \leq n$.*

Proof. Recall that $i_\mu j_\lambda(Y) = P(a)^{\oplus m}$ for some integer m . If $m = 0$, then $i_\mu j_\lambda(Y) = 0$, so $\alpha_{P(a), j_\lambda(Y)}(j_\lambda(g_i)) = 0$ for any $1 \leq i \leq n$, which implies that $j_\lambda(g_i) = 0$, and then $g_i = 0$ for any $1 \leq i \leq n$.

If $m > 0$, then $\alpha_{X, j_\lambda(Y)}(j_\lambda(g_i)i_\lambda(f_i)) = \alpha_{P(a), j_\lambda(Y)}(j_\lambda(g_i))f_i$ for any $1 \leq i \leq n$. Since $i_\mu j_\lambda(Y) = P(a)^{\oplus m}$, from (2) and the section map $\mu_{P(a)}$, we get that $\alpha_{P(a), j_\lambda(Y)}(j_\lambda(g_i))f_i = i_\mu j_\lambda(g_i)\mu_{P(a)}f_i$ is of form

$$X \xrightarrow{f_i} P(a) \xrightarrow{\begin{pmatrix} k_{i1} \\ \vdots \\ k_{im} \end{pmatrix}} P(a)^{\oplus m},$$

where $k_{ij} \in K$ for any $1 \leq j \leq m$.

On the other hand, $\alpha_{X, j_\lambda(Y)}(\sum_{i=1}^n j_\lambda(g_i)i_\lambda(f_i)) = \sum_{i=1}^n \alpha_{P(a), j_\lambda(Y)}(j_\lambda(g_i))f_i$, which is equal to

$$\sum_{i=1}^n \begin{pmatrix} k_{i1} \\ \vdots \\ k_{im} \end{pmatrix} f_i = 0.$$

Since f_1, \dots, f_n are linearly independent, we get that $k_{ij} = 0$ for all $1 \leq i \leq n$, $1 \leq j \leq m$. Therefore, $\alpha_{P(a), j_\lambda(Y)}(g_i) = 0$ for any $1 \leq i \leq n$, and then $g_i = 0$ for any $1 \leq i \leq n$. \square

Similarly, we get the following lemma.

Lemma 4.6. *Let $\Lambda = KQ_\Lambda/I_\Lambda$ be a simple gluing algebra of the two finite-dimensional bound quiver algebras $A = KQ_A/I_A$ and $B = KQ_B/I_B$ by identifying $a \in Q_A$ and $b \in Q_B$. Let $X \in \text{Gproj } A$ and $Y \in \text{Gproj } B$ be indecomposable Gorenstein projective modules. For any morphisms $f_i : P(a) \rightarrow X$ and $g_i : Y \rightarrow Q(b)$ where $1 \leq i \leq n$, if g_1, \dots, g_n are linearly independent and $\sum_{i=1}^n i_\lambda(f_i)j_\lambda(g_i) = 0$, then $f_i = 0$ for any $1 \leq i \leq n$.*

Remark 4.7. Let $X \in \text{Gproj } A$ and $Y \in \text{Gproj } B$ be indecomposable Gorenstein projective modules. Recall that $i_\mu i_\lambda(X) = X \oplus P(a)^{\oplus s}$ and $i_\mu j_\lambda(Y) = P(b)^{\oplus t}$ for some integers s, t . For any morphisms $f : P(a)^{\oplus n} \rightarrow X$ and $g : Y \rightarrow Q(b)^{\oplus n}$ with $\text{Im } g \subseteq \text{rad } Q(b)^{\oplus n}$, we have that $i_\mu(i_\lambda(f)j_\lambda(g)) : P(b)^{\oplus t} \rightarrow X \oplus P(a)^{\oplus s}$ is of form $\begin{pmatrix} 0 \\ h \end{pmatrix}$, where h is a $s \times t$ matrix with entries in K .

Proof. First, $i_\mu(U(v)^{\oplus n}) = P(a)^{\oplus n} \oplus P(a)^{\oplus r}$ for some integer r . From (1) and (2), we get that $i_\mu j_\lambda(g) : i_\mu j_\lambda Y = P(a)^{\oplus t} \rightarrow i_\mu(U(v)^{\oplus n}) = P(a)^{\oplus n} \oplus P(a)^{\oplus r}$ is of form $\begin{pmatrix} 0 \\ h_2 \end{pmatrix}$ where h_2 is a $r \times t$ matrix with entries in K , since $\text{Im } g \subseteq \text{rad } Q(b)^{\oplus n}$, and $i_\mu i_\lambda(f) : i_\mu(U(v)^{\oplus n}) = P(a)^{\oplus n} \oplus P(a)^{\oplus r} \rightarrow X \oplus P(a)^{\oplus s}$ is of form $\begin{pmatrix} f \\ h_1 \end{pmatrix}$, where h_1 is a $s \times r$ matrix with entries in K . Then $i_\mu(i_\lambda(f)j_\lambda(g)) = \begin{pmatrix} 0 \\ h_1 h_2 \end{pmatrix}$ where $h_2 h_1$ is a $s \times t$ matrix with entries in K . \square

Now, we get the final main result in this paper.

Theorem 4.8. Let Λ be a simple gluing algebra of the two finite-dimensional bound quiver algebras $A = KQ_A/I_A$ and $B = KQ_B/I_B$ by identifying $a \in Q_A$ and $b \in Q_B$. Then Λ is CM-finite if and only if A and B are CM-finite. In this case, the Cohen-Macaulay Auslander algebra $\text{Aus}(\text{Gproj}(\Lambda))$ is a simple gluing algebra of $\text{Aus}(\text{Gproj}(A))$ and $\text{Aus}(\text{Gproj}(B))$ by identifying the vertices corresponding to the indecomposable projective modules $P(a)$ and $Q(b)$.

Proof. From Proposition 4.4, the Auslander-Reiten quiver Q_Λ^{Aus} of Λ is a simple gluing quiver of the Auslander-Reiten quiver Q_A^{Aus} of A and the Auslander-Reiten quiver Q_B^{Aus} of B by identifying the vertices corresponding to the indecomposable projective modules $P(a)$ and $Q(b)$.

On the other hand, from the above, we also get that all the irreducible morphisms in $\text{Gproj } \Lambda$ are induced explicitly by the ones in $\text{Gproj } A$ and $\text{Gproj } B$. So for any irreducible morphism in $\text{Gproj } \Lambda$, it is either of form $i_\lambda(f)$ for some irreducible morphism f in $\text{Gproj } A$ or of form $j_\lambda(g)$ for some irreducible morphism g in $\text{Gproj } B$. By viewing Q_A^{Aus} and Q_B^{Aus} to be subquivers of Q_Λ^{Aus} , we identify f with $i_\lambda(f)$, g with $j_\lambda(g)$ for any irreducible morphisms f in $\text{Gproj } A$, and g in $\text{Gproj } B$. Let $\text{Aus}(\text{Gproj } A)$, $\text{Aus}(\text{Gproj } B)$ and $\text{Aus}(\text{Gproj } \Lambda)$ be the opposite algebras of $KQ_A^{\text{Aus}}/I_A^{\text{Aus}}$, $KQ_B^{\text{Aus}}/I_B^{\text{Aus}}$ and $KQ_\Lambda^{\text{Aus}}/I_\Lambda^{\text{Aus}}$ respectively. In this way, we claim that $I_\Lambda^{\text{Aus}} = \langle I_A^{\text{Aus}}, I_B^{\text{Aus}} \rangle$.

First, it is easy to see that $I_A^{\text{Aus}} \subseteq I_\Lambda^{\text{Aus}}$, and $I_B^{\text{Aus}} \subseteq I_\Lambda^{\text{Aus}}$, which implies that $\langle I_A^{\text{Aus}}, I_B^{\text{Aus}} \rangle \subseteq I_\Lambda^{\text{Aus}}$.

Second, for any element $w \in KQ_\Lambda^{\text{Aus}}$ which is in I_Λ^{Aus} , without loss of generality, we assume that w starts from an indecomposable Gorenstein projective Λ -module Z_0 , and ends to an indecomposable Gorenstein projective Λ -module Z_1 . It is easy to see that w is a linear combination of combinations of irreducible morphisms in $\text{Gproj } \Lambda$ by viewing arrows to be irreducible morphisms. Recall that the Auslander-Reiten quiver Q_Λ^{Aus} is a simple gluing quiver of the Auslander-Reiten quivers Q_A^{Aus} and Q_B^{Aus} by identifying the vertices corresponding to the indecomposable projective modules $P(a)$ and $Q(b)$. The proof can be broken into the following four cases.

Case (a) $Z_0 = i_\lambda(X)$, $Z_1 = j_\lambda(Y)$ for some $X \in \text{Ind Gproj } A$, $Y \in \text{Ind Gproj } B$, and w is of form

$$\begin{aligned} i_\lambda(X) &\xrightarrow{i_\lambda(f_0)} U(v)^{\oplus n_0} \xrightarrow{j_\lambda(g_0)} U(v)^{\oplus m_0} \xrightarrow{i_\lambda(f_1)} U(v)^{\oplus n_1} \xrightarrow{j_\lambda(g_1)} U(v)^{\oplus m_1} \\ &\xrightarrow{i_\lambda(f_2)} \dots \xrightarrow{j_\lambda(g_{t-1})} U(v)^{\oplus m_{t-1}} \xrightarrow{i_\lambda(f_t)} U(v)^{\oplus n_t} \xrightarrow{j_\lambda(g_t)} j_\lambda(X_t) = j_\lambda(Y), \end{aligned}$$

with $\text{Im}(f_i) \subseteq \text{rad } P(a)^{\oplus n_i}$ for any $0 \leq i \leq t$ and $\text{Im}(g_j) \subseteq \text{rad } Q(b)^{\oplus m_j}$ for any $0 \leq j \leq t-1$. We prove that $w \in \langle I_A^{\text{Aus}}, I_B^{\text{Aus}} \rangle$ by induction on t .

If $t = 0$, then it follows from Lemma 4.5. For $t > 0$, let f be of form

$$X \xrightarrow{\begin{pmatrix} f_{01} \\ \vdots \\ f_{0n_0} \end{pmatrix}} P(a)^{\oplus n_0}.$$

If $f_{01} = 0, \dots, f_{0n_0} = 0$, then $f_0 = 0$, which implies that $w \in \langle I_A^{\text{Aus}} \rangle \subseteq \langle I_A^{\text{Aus}}, I_B^{\text{Aus}} \rangle$. Otherwise, without loss of generality, we assume that $f_{01}, f_{02}, \dots, f_{0n_0}$ are linearly independent. Then w is of form

$$i_\lambda(X) \xrightarrow{\begin{pmatrix} i_\lambda(f_{01}) \\ \vdots \\ i_\lambda(f_{0n_0}) \end{pmatrix}} U(v)^{\oplus n_0} \xrightarrow{(l_1, l_2, \dots, l_{n_0})} j_\lambda(Y),$$

where $(l_1, l_2, \dots, l_{n_0}) = j_\lambda(g_t) \cdots i_\lambda(f_1) j_\lambda(g_0)$. Furthermore,

$$\alpha_{X, j_\lambda(Y)} \left(\sum_{i=1}^{n_0} l_i i_\lambda(f_{0i}) \right) = \sum_{i=1}^{n_0} i_\mu(l_i) \mu_{P(a)^{\oplus n_0}} f_{0i}.$$

Since $\text{Im}(g_j) \subseteq \text{rad } Q(b)^{\oplus m_j}$ for any $0 \leq j \leq t-1$, from Remark 4.7 and (2), we get that $i_\mu(l_i) : P(a)^{\oplus s} \rightarrow P(a)^m$ is represented by a matrix with its entries in K , which implies that this property holds for $i_\mu(l_i) \mu_{P(a)^{\oplus n_0}}$ by the form of the section map $\mu_{P(a)^{\oplus n_0}}$. Similar to the proof of Lemma 4.5, we get that $i_\mu(l_i) \mu_{P(a)^{\oplus n_0}} = 0$ for each $1 \leq i \leq n_0$, and then $j_\lambda(g_t) \cdots i_\lambda(f_1) j_\lambda(g_0) = 0$.

In order to get that $j_\lambda(g_t) \cdots i_\lambda(f_1) j_\lambda(g_0) \in \langle I_A^{\text{Aus}}, I_B^{\text{Aus}} \rangle$, without loss of generality, we assume $n_0 = 1$. Then g_0 is of form

$$Q(b) \xrightarrow{\begin{pmatrix} g_{01} \\ \vdots \\ g_{0m_0} \end{pmatrix}} Q(b)^{\oplus m_0}.$$

Similar to the above, it is enough to prove that for the case when g_{01}, \dots, g_{0m_0} are linear independent. Let $(p_1, \dots, p_{m_0}) : U(v)^{\oplus m_0} \rightarrow j_\lambda(Y)$ be the morphism $j_\lambda(g_t) \cdots i_\lambda(f_1)$. Then

$$\begin{aligned} & \beta_{Q(b), j_\lambda(Y)}(j_\lambda(g_t) \cdots i_\lambda(f_1) j_\lambda(g_0)) \\ &= j_\mu(j_\lambda(g_t) \cdots i_\lambda(f_1)) \nu_{Q(b)^{\oplus m_0}} g_0 = 0. \end{aligned}$$

Similar to the proof of Remark 4.7, since $j_\mu j_\lambda(Y) = Y \oplus Q(b)^{\oplus q}$ for some integer q , we get that $j_\mu(j_\lambda(g_t) \cdots i_\lambda(f_1)) \nu_{Q(b)^{\oplus m_0}} : Q(b)^{\oplus m_0} \rightarrow Y \oplus Q(b)^{\oplus q}$ is of form $\begin{pmatrix} 0 \\ h' \end{pmatrix}$ where h' is a $q \times m_0$ matrix with entries in K . Since g_{01}, \dots, g_{0m_0} are linear independent,

$$\beta_{Q(b), j_\lambda(Y)}(j_\lambda(g_t) \cdots i_\lambda(f_1)) = j_\mu(j_\lambda(g_t) \cdots i_\lambda(f_1)) \nu_{Q(b)^{\oplus m_0}} = 0$$

and then $j_\lambda(g_t) \cdots i_\lambda(f_1) = 0$. From the assumption of induction, we have

$$j_\lambda(g_t) i_\lambda(f_t) \cdots j_\lambda(g_1) i_\lambda(f_1) \in \langle I_A^{\text{Aus}}, I_B^{\text{Aus}} \rangle,$$

and then $w \in \langle I_A^{\text{Aus}}, I_B^{\text{Aus}} \rangle$.

Case (b) $Z_0 = j_\lambda(Y)$ and $Z_1 = i_\lambda(X)$ for some $X \in \text{Ind Gproj } A$ and $Y \in \text{Ind Gproj } B$. $Z_0 = i_\lambda(X)$, $Z_1 = j_\lambda(Y)$ for some $X \in \text{Ind Gproj } A$, $Y \in \text{Ind Gproj } B$, and w is of form

$$\begin{aligned} j_\lambda(Y) &\xrightarrow{j_\lambda(f_0)} U(v)^{\oplus n_0} \xrightarrow{i_\lambda(g_0)} U(v)^{\oplus m_0} \xrightarrow{j_\lambda(f_1)} U(v)^{\oplus n_1} \xrightarrow{i_\lambda(g_1)} U(v)^{\oplus m_1} \\ &\xrightarrow{j_\lambda(f_2)} \dots \xrightarrow{i_\lambda(g_{t-1})} U(v)^{\oplus m_{t-1}} \xrightarrow{j_\lambda(f_t)} U(v)^{\oplus n_t} \xrightarrow{i_\lambda(g_t)} i_\lambda(Y_t) = i_\lambda(X), \end{aligned}$$

with $\text{Im}(f_i) \subseteq \text{rad } P(a)^{\oplus n_i}$ for any $0 \leq i \leq t$ and $\text{Im}(g_j) \subseteq \text{rad } Q(b)^{\oplus m_j}$ for any $0 \leq j \leq t-1$. It is similar to Case (a).

Case (c) $Z_0 = i_\lambda(X)$, $Z_1 = i_\lambda(Y)$ for some $X, Y \in \text{Ind Gproj } A$, and w is of form

$$\begin{aligned} i_\lambda(X) &\xrightarrow{i_\lambda(f_0)} U(v)^{\oplus n_0} \xrightarrow{j_\lambda(g_0)} U(v)^{\oplus m_0} \xrightarrow{i_\lambda(f_1)} U(v)^{\oplus n_1} \xrightarrow{j_\lambda(g_1)} U(v)^{\oplus m_1} \\ &\xrightarrow{i_\lambda(f_2)} \dots \xrightarrow{j_\lambda(g_{t-1})} U(v)^{\oplus m_{t-1}} \xrightarrow{i_\lambda(f_t)} i_\lambda(X_t) = i_\lambda(Y), \end{aligned}$$

with $\text{Im}(f_i) \subseteq \text{rad } P(a)^{\oplus n_i}$ for any $0 \leq i \leq t-1$ and $\text{Im}(g_j) \subseteq \text{rad } Q(b)^{\oplus m_j}$ for any $0 \leq j \leq t-1$. If $t = 0$, then $i_\lambda(f_0) = 0$ and then $f_0 = 0$, which implies that $w \in I_A^{\text{Aus}}$. For $t > 0$, we can prove it similar to Case (a).

Case (d) $Z_0 = j_\lambda(X)$, $Z_1 = j_\lambda(Y)$ for some $X, Y \in \text{Ind Gproj } B$, and w is of form

$$\begin{aligned} j_\lambda(X) &\xrightarrow{j_\lambda(f_0)} U(v)^{\oplus n_0} \xrightarrow{i_\lambda(g_0)} U(v)^{\oplus m_0} \xrightarrow{j_\lambda(f_1)} U(v)^{\oplus n_1} \xrightarrow{i_\lambda(g_1)} U(v)^{\oplus m_1} \\ &\xrightarrow{j_\lambda(f_2)} \dots \xrightarrow{i_\lambda(g_{t-1})} U(v)^{\oplus m_{t-1}} \xrightarrow{j_\lambda(f_t)} j_\lambda(X_t) = j_\lambda(Y), \end{aligned}$$

with $\text{Im}(f_i) \subseteq \text{rad } P(a)^{\oplus n_i}$ for any $0 \leq i \leq t-1$ and $\text{Im}(g_j) \subseteq \text{rad } Q(b)^{\oplus m_j}$ for any $0 \leq j \leq t-1$. It is similar to Case (c).

To sum up, $KQ_\Lambda^{\text{Aus}}/I^{\text{Aus}}$ is a simple gluing algebra of $KQ_A^{\text{Aus}}/I_A^{\text{Aus}}$ and $KQ_B^{\text{Aus}}/I_B^{\text{Aus}}$ by identifying the vertices corresponding to the indecomposable projective modules $P(a)$ and $Q(b)$, and then $\text{Aus}(\text{Gproj}(\Lambda))$ is a simple gluing algebra of $\text{Aus}(\text{Gproj}(A))$ and $\text{Aus}(\text{Gproj}(B))$ by identifying the vertices corresponding to the indecomposable projective modules $P(a), Q(b)$. \square

Example 4.9. Let $\Lambda = KQ/I$ be the algebra where Q is the quiver as the left quiver in Figure 4 shows, and $I = \langle \alpha_{i+1}\alpha_i, \varepsilon^2 | i \in \mathbb{Z}/3\mathbb{Z} \rangle$. Let A be the quotient algebra $\Lambda/\langle \varepsilon \rangle$, and $B = e_3\Lambda e_3$. Then Λ is a simple gluing algebra of A, B . Let $\text{Aus}(\text{Gproj}(\Lambda))$ be the algebra corresponding to the bound quiver $(Q^{\text{Aus}}, I^{\text{Aus}})$. Then Q^{Aus} is as the right quiver in Figure 4 shows, and $I^{\text{Aus}} = \langle \alpha_{i+1}^+\alpha_i^-, \varepsilon^+\varepsilon^- | i \in \mathbb{Z}/3\mathbb{Z} \rangle$. It is easy to see that $\text{Aus}(\text{Gproj}(\Lambda))$ is the simple gluing algebra of $\text{Aus}(\text{Gproj}(A))$ and $\text{Aus}(\text{Gproj}(B))$ by identifying the vertices corresponding to the indecomposable projective modules $P(3)$ and $Q(3)$.

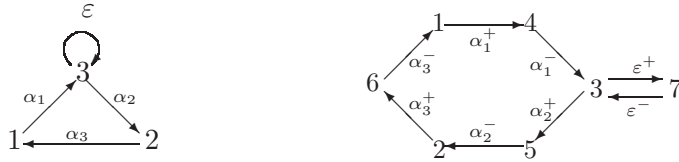


Figure 4. The quiver of Λ and its Cohen-Macaulay Auslander algebra.

Example 4.10. Following Example 3.19, let $\Lambda = KQ/I$ be the algebra where Q is the quiver

$$1 \xrightleftharpoons[\beta_1]{\alpha_1} 2 \xrightleftharpoons[\beta_2]{\alpha_2} 3 \xleftarrow{\gamma} 4, \text{ and } I = \langle \beta_i\alpha_i, \alpha_i\beta_i, \beta_2\gamma | i = 1, 2 \rangle. \text{ Let } \text{Aus}(\text{Gproj}(\Lambda)) \text{ be the algebra}$$

corresponding to the bound quiver $(Q^{\text{Aus}}, I^{\text{Aus}})$. Then Q^{Aus} is as Figure 5 shows, and $I^{\text{Aus}} = \langle \alpha_1^+\beta_1^-, \beta_1^-\alpha_1^+, \alpha_2\beta_2, \beta_2\alpha_2, \beta_2\gamma \rangle$. Let A and B be as in Example 3.19. It is easy to see that

$\text{Aus}(\text{Gproj}(\Lambda))$ is the simple gluing algebra of $\text{Aus}(\text{Gproj}(A))$ and $\text{Aus}(\text{Gproj}(B))$ by identifying the vertices corresponding to the indecomposable projective modules $P(2)$ and $Q(2)$.

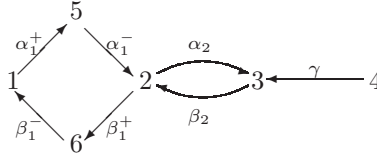


Figure 5. Cohen-Macaulay Auslander algebra of Λ .

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